

IMPROVED BOUNDS FOR RELAXED GRACEFUL TREES

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ABSTRACT. We introduce left and right-layered trees as trees with a specific representation and define the excess of a tree. Applying these ideas, we show a range-relaxed graceful labeling which improves on the upper bound for maximum vertex label given by Van Bussel [11]. For the case when the tree is a lobster of size m and diameter d , the labeling produces vertex labels no greater than $\frac{3}{2}m - \frac{1}{2}d$. Furthermore, we show that any lobster T with m edges and diameter d has an edge-relaxed graceful bipartite labeling with at least $\max\{\frac{3m-d+6}{4}, \frac{5m+d+3}{8}\}$ of the edge weights distinct, which is an improvement on a bound given by Rosa and Širáň [9] on the α -size of trees, for $d < \frac{m+22}{7}$ and $d > \frac{5m+19}{7}$. We also show that there exists an edge-relaxed graceful labeling (not necessarily bipartite) with at least $\max\{\frac{3}{4}m + \frac{d-\nu}{8} + \frac{3}{2}, \nu\}$ of the edge weights distinct, where ν is twice the size of a partial matching of T . This is an improvement on the gracesize bound from [9] for certain values of ν and d . We view these results as a step towards Bermond's conjecture [1].

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1. INTRODUCTION

The graceful tree conjecture (GTC), first formulated in 1966 by Alexander Rosa [8], has played a role as the point of origin for most of the questions and results in graph labeling. At this time, Joe Gallian's Dynamic Survey of Graph Labeling shows one-thousand-four-hundred-and-sixty references [6] most of which can claim the GTC as its root.

The question, as with many popular open combinatorial problems, is easy to state.

Let T be a tree on n vertices. Define a weight on an edge as the absolute difference of the labels of its incident vertices.

Conjecture 1.1. [8] *It is possible to label the vertices of T uniquely from 0 to $n - 1$, so that the set of weights of T is $\{1, 2, \dots, n - 1\}$.*

Every tree that accepts the conjectured labeling is known as a *graceful tree*. Though many labeling schemes exist to prove specific families of trees graceful, there has been little progress in proving the conjecture even for a robust class of “shallow” trees. More precisely, using a definition of *tree distance* that first appeared in [7], let P be a longest path in T and call T a k -distant tree if all of its vertices are a distance at most k from P . Paths

(0-distant trees) and caterpillars (1-distant trees) were shown to be graceful in [8]. However, the conjecture is unknown for any trees with higher tree distance. In fact, the problem for 3-distant trees, or *lobsters*, is a well-known conjecture of Jean-Claude Bermond.

Conjecture 1.2. [1] *Every lobster is graceful.*

One approximate approach to the GTC, introduced by Solomon Golomb in the 1970's, is the following: Let G be a graph with vertex set V and edge set E , $f(V) \rightarrow \mathbb{N}$ an injective map to $\{1, \dots, m'\}$ for some $m' \geq m = |E|$ (producing the vertex labels), and $g(E) \rightarrow \mathbb{N}$ an injective map to $\{1, \dots, m''\}$ for some $m'' \geq m = |E|$ defined by $g(uv) = |f(u) - f(v)|$ (producing the induced edge weights under f). The map f is called a *range-relaxed labeling*.

The “best” bound on the maximum vertex labels in a range-relaxed labeling of trees was given by Frank Van Bussel [11].

Theorem 1.3 (Van Bussel). *Every tree T on m edges has a range-relaxed graceful labeling f with vertex labels in the range, $0, \dots, 2m - \text{diameter}(T)$.*

Another relaxation of the graceful condition was defined by Alexander Rosa and Jozef Širáň [9]. We first define terms in the language of [11]. Let G be a graph with vertex set V and edge set E , $f(V) \rightarrow \mathbb{N}$ an injective map to $\{1, \dots, m\}$ where $m = |E|$ (producing the vertex labels), and $g(E) \rightarrow \mathbb{N}$ a map to $\{1, \dots, m\}$ defined by $g(uv) = |f(u) - f(v)|$ (producing the induced edge weights under f). Such a map f is called an *edge-relaxed labeling*.

Theorem 1.4 (Rosa-Širáň). *Every tree with m edges has an edge-relaxed graceful labeling with at least $\frac{5}{7}(m+1)$ of the edge weights distinct.*

Call the largest number of distinct edge weights in an edge-relaxed labeling of a tree T , the *gracesize* of T , $gs(T)$. With this notation, the above theorem presents the lower bound $gs(T) \geq \frac{5}{7}(m+1)$.

We call a labeling f *bipartite*, if there exists an integer c such that for any edge uv , either $f(u) \leq c < f(v)$ or $f(v) \leq c < f(u)$. A bipartite labeling that is graceful is called an α -*labeling*. As with the concept of gracesize, we can let the α -*size* of a tree T , $\alpha(T)$, be the largest number of distinct edge weights in any bipartite labeling of T . For a given bipartite labeling f of T , we write $\varepsilon(f)$ for the cardinality of the set of weights on the edges of T induced by f .

The authors of [9] proved that for any tree T with m edges, $\frac{5}{7}(m+1) \leq \alpha(T) \leq \frac{5}{6}(m+9)$.

It should be noted that the lower bound for gracesize was taken from that for α -size, as it was not known how to use non-bipartite labelings to improve this bound.

Several results have come out in the intervening years on the lower bounds for the α -size of certain restricted families of trees. For the case of trees with maximum degree 3, C. Paul Bonnington and Jozef Širáň [2] showed that $\alpha(T) \geq \frac{5}{6}(m+1)$. Ljiljana Brankovic, Alexander Rosa, and Jozef Širáň [3] later improved this bound to $\alpha(T) \geq \lfloor \frac{6}{7}(m+1) \rfloor - 1$.

A direction with some positive advancement has been for trees that have a perfect matching. In 1999, Hajo Broersma and Cornelis Hoede [4] showed

a surprising equivalence between the GTC and a more restrictive labeling on trees containing a perfect matching.

Let T be a tree of order n with a perfect matching M . A graceful labeling of T , which additionally satisfies the property that for any edge in M , the pair of vertices incident to that edge must have a label sum of $n - 1$, is called a *strongly graceful* labeling. For any tree T with a perfect matching M , the tree resulting from the contraction of the edges of M is called the *contree* of T .

Theorem 1.5 (Broersma-Hoede). [4] *Every tree is graceful if and only if every tree containing a perfect matching is strongly graceful.*

Furthermore, the authors proved

Corollary 1.6 (Broersma-Hoede). [4] *Every tree containing a perfect matching and having a caterpillar for its contree is strongly graceful.*

Although the following theorem is easily implied by the proof of Theorem 1.5, and is an immediate consequence of the above corollary, the authors did not state it. David Morgan proved the following by an explicit construction.

Theorem 1.7 (Morgan). [7] *All lobsters with perfect matchings are graceful.*

Our result requires the flexibility of the labeling in the proof of Theorem 1.5 and Corollary 1.6. We will call the graceful labelings of lobsters with a perfect matching described in [4], the *Broersma-Hoede labeling*, or *BH labeling*. For completeness, we review this labeling in section 2.2.

In this paper, we introduce the concept of the *excess* of a tree and use it to improve the bounds on the ranges for both range-relaxed and edge-relaxed graceful labelings when the tree in question is a lobster. For the latter result, we use BH labelings to slightly improve the gracesize bound obtained from analyzing the excess.

For basic graph theoretic concepts, we refer the reader to The Book [12].

2. THE EXCESS OF LAYERED TREES

Let T be a rooted tree. For any vertex $v \in V(T)$, let $\gamma(v)$ denote the number of levels in T where v has at least one descendant.

Our labeling applies to rooted trees where the root r has been chosen in such a way that $\deg(r) = 1$ and $\gamma(r) = d = \text{diameter}(T)$. We will order the vertices within each level according to their degrees and the associated parameter γ so that edges do not cross. The level of the root vertex r is denoted level zero, L_0 , and vertices of distance $k > 0$ from r are on level k , or L_k , represented k levels below r . We denote by $u \prec v$ the placement of u to the left of v . With this notation, we define the order on each level.

- (1) If u and v are siblings of degree one, the order of u and v is arbitrary.
- (2) If u and v are siblings and $\gamma(u) < \gamma(v)$, then $u \prec v$.
- (3) If u and v are siblings and $\gamma(u) = \gamma(v)$, and $\deg(u) \geq \deg(v)$, then $u \prec v$.
- (4) If u and v are siblings and $u \prec v$, a and b descendants of u and v respectively, on the same level, then $a \prec b$.

We call a rooted tree so represented a *left-layered tree*. By this ordering, a path of maximum length is drawn on the right extreme of the picture.

Let L_j denote the set of vertices of T on level j . That is, $L_j = \{v_i^j : 1 \leq i \leq n_j\}$ where $n_j = |L_j|$. We assume that $v_i^j \prec v_{i+1}^j$ for all $1 \leq i \leq n_j - 1$. Let i be the smallest index such that $\text{dist}(v_i^j, v_{i+1}^j) > 2$. We define ex_j to be the cardinality of the set $\{v \in L_{j-1} : u \prec v\}$ where u is the parent of v_i^j . That is, ex_j counts the number of consecutive vertices on level j with distance greater than 2 along with the number of siblings linearly between their parents (aunts and uncles). Define the *excess* of T , denoted by $ex(T)$, as

$$ex(T) = \sum_{j=0}^d ex_j$$

Notice that if $|L_j| = 1$, $ex_j = 0$. If $|L_j| > 1$ and $\text{dist}(v_a^j, v_b^j) = 2$, for every pair of vertices in L_j , $ex_j = 0$.

We define ex'_j to be the number of consecutive vertices on level j with distance greater than 2 and

$$ex'(T) = \sum_{j=0}^d ex'_j$$

We define a *right-layered tree* representation by the definition of left-layered trees above with rule (3) replaced by the following:

(3') If u and v are siblings on the same level and $\gamma(u) = \gamma(v)$, and $\deg(u) \leq \deg(v)$, then $u \prec v$.

The excess of right-layered trees is defined in the same way as for left-layered trees.

Observation 2.1. *Notice that for lobsters (2-distant trees)*

$$ex'(T) = ex(T) \tag{2.1}$$

Similarly for closer relations, let s_j be the number of consecutive vertices on level j with distance equal to 2. That is, $s_j = n_j - ex_j - 1$. Define the *surplus* of T , denoted by $s(T)$, as

$$s(T) = \sum_{j=3}^d s_j$$

2.1. Range-Relaxed Labelings.

Theorem 2.2. *Every tree T of size m has a range-relaxed graceful labeling with vertex labels in the range $0, \dots, m + ex(T)$.*

Proof. We assign labels on the vertices of left-layered tree T with size m and diameter d .

For $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, assign the labels from the interval

$$\left[\sum_{j=1}^i (n_{d+2-2j} + ex_{d+2-2j}), \sum_{j=1}^i (n_{d+2-2j} + ex_{d+2-2j}) + n_{d-2i} - 1 \right]$$

to the vertices of L_{d-2i} .

We will use the parameter

$$B = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} (n_{d-2j} + ex_{d-2j})$$

to define the labels on the remaining levels.

For $1 \leq i \leq \lceil \frac{d}{2} \rceil$, assign the labels from the interval

$$\left[B + \sum_{j=i}^{\lceil \frac{d}{2} \rceil} (n_{d+1-2j} + ex_{d+1-2j}) - n_{d+1-2i}, B + \sum_{j=i}^{\lceil \frac{d}{2} \rceil} (n_{d+1-2j} + ex_{d+1-2j}) - 1 \right]$$

to the vertices of L_{d+1-2i} .

On levels $L_d, L_{d-2}, L_{d-4}, \dots$, assign labels from left to right in descending order, and on levels $L_{d-1}, L_{d-3}, L_{d-5}, \dots$, assign labels from left to right in ascending order. Since our representation of T does not allow crossing of edges, there are no repetition of weights.

Notice that the weights on each edge-level are in strictly ascending order. To see this, suppose that $t \in \{1, 2, \dots, d-1\}$. Let $x \in L_{t-1}, y, z \in L_t$, and $w \in L_{t+1}$, such that xy has the largest weight in the edge-level L_t and zw has the smallest weight in the edge-level L_{t+1} . When $x < y$, $x = w + ex_{t+1} + 1$ and $z = y - ex_{t+1}$, implying $z - w = y - ex_{t+1} - (x - ex_{t+1} - 1) = y - x + 1$. When $x > y$, $w = x + ex_{t+1} + 1$ and $z = y + ex_{t+1}$, implying $w - z = x + ex_{t+1} + 1 - (y + ex_{t+1}) = x - y + 1$. Therefore, the edges xy and zw have consecutive weights, which means that the weights induced by this assignment are all distinct.

Since the largest label is

$$\begin{aligned} & B - 1 + \sum_{j=1}^{\lceil \frac{d}{2} \rceil} (n_{d+1-2j} + ex_{d+1-2j}) \\ &= \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} n_{d-2j} + \sum_{j=1}^{\lceil \frac{d}{2} \rceil} n_{d+1-2j} - 1 + \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} ex_{d-2j} + \sum_{j=1}^{\lceil \frac{d}{2} \rceil} ex_{d+1-2j} \\ &= \sum_{j=0}^d n_j - 1 + \sum_{j=0}^d ex_j \\ &= m + ex(T). \end{aligned}$$

Therefore, the assignment is a range-relaxed graceful labeling of T . \square

Corollary 2.3 (Rosa). [8] *Caterpillars are graceful.*

Proof. Notice that for any caterpillar C , $ex(C) = 0$. \square

Theorem 2.4. *If T is a lobster of size m and diameter d , then $v_{max} \leq \frac{3}{2}m - \frac{1}{2}d$.*

Proof. Notice that for any 2-distant tree T ,

$$s(T) + ex'(T) + d = m \tag{2.2}$$

and since any pair of consecutive vertices on a given level with distance greater than 2 must be preceded by ancestors one level above, whose distance is 2,

$$s(T) \geq ex'(T). \quad (2.3)$$

We apply 2.2 and 2.3 to the maximum vertex label given by Theorem 2.2. \square

2.2. Lobster Shells. Next, we prepare a structure which, though interesting in its own right, will be used to apply BH labelings to edge-relaxed labelings.

Definition 2.5. *A lobster T is a shell (sometimes called a lobster shell) if there exists a longest path P , so that all vertices not on P with a neighbor on P have degree two.*

For any right-layered represented lobster T with longest path P , the shell of T is found by contracting all but one leaf vertex adjacent to vertices of distance one to P , and contracting all leaf vertices adjacent to P .

Proposition 2.6. *For any lobster T of order n , the shell of T has a perfect matching if n is even or a $n-1$ -matching (a matching covering $n-1$ vertices) if n is odd.*

Proof. If the shell of T is a path, then the statement is true. Otherwise, the shell of T is a maximal path incident to some number of branches composed of two edges. The statement is easy to show by induction on the number of such branches. \square

We now review the labeling shown in [4]. For any lobster T of even order n with a perfect matching M , a BH labeling can be found by contracting the edges of M to form the *contree* T' , which is a caterpillar, or 1-distant tree. For any graceful labeling f of T' , label the vertices of T' by $2f$, and then expand the edges back to form T where only half the vertices are labeled such that the edges of T' correspond to the edges of T not in M . To label the remainder of the vertices of T , suppose u is a labeled vertex and notice that u is adjacent to an unlabeled vertex v by an edge from M . Label v by $n - 1 - 2f(u)$.

Proposition 2.7. *For any lobster T of order n and non-negative integer l , if there exists a right-layered representation of T so that the shell of T can be found by l contractions, then*

$$gs(T) \geq \begin{cases} n - l - 1, & \text{if } n - l \text{ is even} \\ n - l - 2, & \text{if } n - l \text{ is odd} \end{cases}$$

Proof. Perform l contractions to produce the shell of T , $S(T)$, and notice by Proposition 2.6 that if $n - l$ is even, then $S(T)$ contains a perfect matching. By Theorem 1.7, $S(T)$ accepts a graceful labeling. Expanding the previously contracted edges and labeling the new vertices uniquely by labels from the set $\{n - l + 1, \dots, n\}$, we create at most l repetitions of edge weights. If $n - l$ is odd, then $S(T)$ contains an $n - 1$ -matching. Call the graph obtained by performing one additional contraction on a leaf of $S(T)$, $S'(T)$. By Theorem 1.7, $S'(T)$ accepts a graceful labeling. Expanding the previously

contracted edges and labeling the new vertices uniquely by labels from the set $\{n-l-2, \dots, n\}$, we create at most $l+1$ repetitions of edge weights. \square

We say that a tree T is *pretty graceful* if there exists a vertex v such that the contraction $T \setminus v$ is graceful, which leads us to the following amusing observation.

Corollary 2.8. *Lobster shells are pretty graceful.*

However, by more careful analysis, we can do a little better. We shall use the following result of Marco Burzio and Giorgio Ferrarese [5].

Definition 2.9. [10] *Let S and T be trees and let u, v be vertices of S and T , respectively. Replace each vertex of S , other than the exceptional vertex u , by a copy of T by identifying each vertex of S with the vertex corresponding to v in the distinct copy of T . Denote the resulting tree by $S\Delta_{+1}T$.*

Theorem 2.10. [10] *If S of order n_S and T are trees with graceful labelings f and g , where $f(u) = n_S - 1$ and $g(v) = 0$, then $S\Delta_{+1}T$ is graceful.*

Suppose S and T have orders n_S and n_T , respectively, with graceful labelings f and g . The construction used to prove the above theorem requires $n_S - 1$ copies of T , each substituted for a vertex of S . Let (A, B) be a bipartition of T with $v \in A$. The labeling function follows, other than the label on u , which is $(n_S - 1)n_T$.

$$g_i(x) = \begin{cases} in_T + g(x), & \text{if } x \in A \\ (n_S - i - 2)n_T + g(x), & \text{if } x \in B \end{cases}$$

Burzio and Ferrarese improved this method and called it the *generalized Δ_{+1} construction* [5] by noticing that for any two adjacent vertices of S into which we substitute copies of T , we may connect two such copies of T by an edge between **any** two vertices that correspond to the same vertex in each copy of T . The exceptional vertex u must still be adjacent to the vertices corresponding to the fixed vertex v of T .

Theorem 2.11. *Lobster shells are graceful.*

Proof. By Proposition 2.6, either a shell has a perfect matching or an almost perfect matching. Suppose the latter. Let P be a path of diameter length. By induction on the number of vertices, it is easy to show that there exists an almost perfect matching M which does not cover an end vertex u of P . However, this means we can apply the generalized Δ_{+1} construction of [5] with u as the exceptional vertex, the caterpillar formed by contracting the edges of M as S (labeled gracefully with largest label on u), and P_2 as T . \square

An *m-comet* is a star with a central vertex, m rays, and m independent leaf edges adjoined to the rays. A comet with some (or no) number of leaf edges adjoined to the central vertex is called a *broken comet* or *stardust*. For any pair of vertex disjoint trees T_1 and T_2 with distinguished vertices u_1 and u_2 , we write $T_1 \circ T_2$ to denote the tree obtained by identifying u_1 with u_2 , also known as an *amalgamation* of T_1 and T_2 .

For any bipartite labeling f of a tree T with m edges, the *complementary labeling* \bar{f} is defined by $\bar{f}(v) = m - f(v)$ for each vertex $v \in T$. Let A be

the part of the bipartition (A, B) of T for which $f^{-1}(0) \in A$. The *inverse labeling* \hat{f} is given by

$$\hat{f}(v) = \begin{cases} |A| - 1 - f(v), & \text{if } v \in A \\ 2|A| + |B| - 1 - f(v), & \text{if } v \in B \end{cases}$$

We will use the following Lemma from [9]

Lemma 2.12 (Rosa-Širáň). [9] *Let T_1 and T_2 be vertex-disjoint trees with distinguished vertices $u_1 \in V(T_1)$ and $u_2 \in V(T_2)$. Assume that there are bipartite labelings f_1 and f_2 of T_1 and T_2 respectively, such that $f_1(u_1) = f_2(u_2) = 0$. Then there exists a bipartite labeling of $T_1 \circ T_2$ such that $\varepsilon(f) \geq \varepsilon(f_1) + \varepsilon(f_2)$. Consequently, for the α -size of $T_1 \circ T_2$ we have $\alpha(T_1 \circ T_2) \geq \varepsilon(f_1) + \varepsilon(f_2)$.*

Theorem 2.13. *Every lobster shell is an amalgamation of stardust. In particular, if T is a lobster shell, then there exist stardust graphs S_1, \dots, S_k with bipartite labelings f_1, \dots, f_k so that $T = S_1 \circ S_2 \circ \dots \circ S_k$ and there exists a bipartite labeling of T so that $\alpha(T) \geq \varepsilon(S_1) + \dots + \varepsilon(S_k)$.*

Proof. We will apply the following labeling from the proof of Lemma 2.12 to compose stardust graphs. We call this labeling the *Rosa-Širáň-labeling* or just *RS-labeling*. For $i = 1, 2$ let (A_i, B_i) be the bipartition of $V(T_i)$ for which $f_i^{-1}(0) = u_i \in A_i$. Define the labeling f of T as

$$f(v) = \begin{cases} \hat{f}_1(v), & \text{if } v \in A_1 \\ \hat{f}_2(v) + |A_1| - 1, & \text{if } v \in A_2 \cup B_2 \\ \hat{f}_1(v) + |A_2| + |B_2| - 1, & \text{if } v \in B_1 \end{cases}$$

If T has diameter 4, then T is a comet and we use the *RS-labeling* on T . Suppose that T has diameter 5. Let P be a path of maximum length and consider a right layered representation of T . Remove the edge on P between L_2 and L_3 , creating two comets. Let T_1 be the comet that contains L_0 and let S_2 be the comet that contains L_5 . Let f be the *RS-labeling* of T_1 and g be the *RS-labeling* of S_2 . Notice that under \hat{f} , the label on the central vertex of T_1 is 0. Thus, we can amalgamate P_2 and T_1 by labeling one vertex, x , of P_2 by 0 and the other, y , by $|T_1|$. Call the amalgamated graph S_1 . Next, notice that y is labeled 0 under $(\hat{f} \circ P_2)$.

We label S_2 by \hat{g} so that the central vertex receives the label 0. Finally, notice that $(\hat{f} \circ P_2) \circ \hat{g}$ is the required labeling of T .

Under this labeling of T , the maximum label is found on L_4 and the label 0 is on L_3 .

If T has diameter 6 (and for each unit increase in diameter) the above labeling can be iterated. Remove the edge on P between L_3 and L_4 , creating a shell of diameter 5, T_1 , and a comet S_2 . Let f be the labeling found for diameter 5 shells and amalgamate P_2 to T_1 with labeling $f \circ P_2$. Let g be the *RS-labeling* of S_2 . We can now amalgamate S_2 to the resulting graph, with labeling $\hat{f} \circ P_2 \circ \hat{g}$. The rest of the proof proceeds similarly as an induction on diameter. □

The α -size of stardust was calculated in Proposition 4 of [9] and allows us to state the following bound.

Corollary 2.14. *For any lobster shell T , $\alpha(T) \geq \lfloor \frac{5m+2}{6} \rfloor$.*

Proof. We use the simple property that for any sequence of non-negative real numbers, x_1, \dots, x_j ,

$$\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \dots + \lfloor x_j \rfloor \geq \left\lfloor \sum_{i=1}^j x_i \right\rfloor + j - 1 \quad (2.4)$$

We decompose T into stardust graphs and a comet as in the above theorem, so that $T = S_1 \circ S_2 \circ \dots \circ S_{k-1} \circ T_k$. Using Proposition 4 of [9], we calculate the α -size of each stardust graph and the comet as

$$\left(\sum_{i=1}^{k-1} \left\lfloor \frac{5m_i + 2}{6} \right\rfloor + 1 \right) + \left\lfloor \frac{5m_k + 2}{6} \right\rfloor$$

where m_i is the size of S_i .

Note that $k = d - 3$, where d is the diameter of T . Applying 2.4 completes the proof. \square

In light of the result from [9] that for any comet T , $\alpha(T) \leq \frac{5}{6}(m + 9)$, the above corollary shows that lobster shells have maximum α -size with respect to the multiplicative constant.

We can extend Definition 2.5 for any three-distant tree T with longest path P by defining the *shell of a three distant tree* as the graph found by contracting all branches not on P with size less than 3.

We challenge the interested reader to prove the following statements.

Conjecture 2.15. *Every lobster with an almost perfect matching is graceful.*

Conjecture 2.16. *Every shell of a three-distant tree is graceful.*

Conjecture 2.17. *Every three-distant tree with a perfect matching is graceful.*

2.3. Edge-Relaxed Labelings. We introduce some useful notation.

- θ_i is the set of those weights of edges on level i that occurred on levels above i (that is, levels $j < i$)
- $\theta = \sum_{i=0}^d \theta_i$
- d_i is the average degree over vertices on level i which are of distance 1 from P and have at least one neighbor not on P
- $s(i, i + 1)$ is the surplus on level i from vertices of distance 1 to P and the surplus on level $i + 1$ of vertices of distance 2 to P
- $\alpha(i, i + 1)$ is the number of distinct weights on levels i and $i + 1$ so that edges on level i are distance 0 or 1 to P and edges on level $i + 1$ are of distance 2 to P
- $m(i, i + 1)$ is the number of edges on levels i and $i + 1$ so that edges on level i are of distance 0 or 1 to P and edges on level $i + 1$ are of distance 2 to P

The following relation is easy to verify

$$s(i, i + 1) + ex_{i+1} + 1 = m(i, i + 1) \quad (2.5)$$

Theorem 2.18. *For any lobster T with m edges and diameter d , $\alpha(T) \geq \max\{\frac{3m-d+6}{4}, \frac{5m+d+3}{8}\}$.*

Proof. We label the right-layered tree T as follows. For odd d , we label vertices consecutively from right to left starting from level d and decreasing levels by one until level 1, from the interval $[0, l = \sum_{i \text{ odd}} |L_i|]$. Then, from level 0 to level $d - 1$, label vertices consecutively from left to right from the interval $[l + 1, \sum_{i=0}^d |L_i|]$.

For even d , we label the vertices consecutively from right to left starting from level d and decreasing levels by one until level 0, from the interval $[0, l = \sum_{i \text{ even}} |L_i|]$. Then, from level 1 to level $d - 1$, label vertices consecutively from left to right from the interval $[l + 1, \sum_{i=0}^d |L_i|]$.

Claim 2.19. *Every edge weight of T may be repeated at most once and only on consecutive levels.*

Proof. Let P be a maximum path of the right-layered representation of T beginning at the root with vertices x_0, x_1, \dots, x_d . Call the above labeling function f , and for any level $L_i, 0 \leq i \leq d$, let $f(L_i)$ denote the labels of vertices on L_i . We say an edge e is *on level i* if e joins vertices on levels $i - 1$ and i .

Notice that by definition of f , $f(L_i)$ has no repetitions for any i .

Let $l_i = |L_i|$ and choose i, j so that $1 \leq i + 1 < j \leq d$. We show that no weight of an edge from L_i can be repeated on L_j . Assume i and j are of the same parity and that $f(x_i)$ is the minimum label on level i . Notice that

$$f(L_i) = [f(x_i), f(x_i) + l_i - 1],$$

$$f(L_j) = [f(x_j), f(x_j) + l_j - 1]$$

and $f(x_i) > f(x_j) + l_j - 1$. Similarly, if $f(x_i)$ is the maximum label on level i , we have

$$f(L_i) = [f(x_i) - l_i + 1, f(x_i)],$$

$$f(L_j) = [f(x_j) - l_j + 1, f(x_j)]$$

and $f(x_i) < f(x_j) + l_j - 1$.

In either case, by considering $f(L_{i-1})$ and $f(L_{j-1})$, it is easy to see that the edge weights on L_i and L_j are distinct.

If i and j are of opposite parity, assume without loss of generality that $f(x_i)$ is the minimum label on level i . Notice that

$$f(L_i) = [f(x_i), f(x_i) + l_i - 1],$$

$$f(L_{j-1}) = [f(x_{j-1}), f(x_{j-1}) + l_{j-1} - 1]$$

and $f(x_i) > f(x_{j-1}) + l_{j-1} - 1$. Again, considering $f(L_{i-1})$ and $f(L_j)$, we see that the weights on L_i and L_j are distinct.

Suppose next that $0 \leq i \leq d - 1$ and consider the weights on L_i and L_{i+1} . Since no edge weight can be repeated on a given level, any edge weight on level L_i can be repeated at most once on level L_{i+1} , which proves the claim. \square

Claim 2.20. $s(i, i + 1) \geq (d_i - 1) \times ex_{i+1}$

Proof. If a pair of vertices u, v on level i with distance 0 or 1 to P contribute to the surplus on level i , and each has a neighbor on L_{i+1} , then u and v have neighbors which contribute to the excess of L_{i+1} . Furthermore, the neighbors of u on L_{i+1} contribute to the surplus on L_{i+1} . \square

Combining 2.5 with the above claim produces

$$d_i \times ex_{i+1} \leq m(i, i+1) - 1 \quad (2.6)$$

Claim 2.21. $\theta_i \leq \left\lceil \frac{d_i-1}{d_i} ex_{i+1} \right\rceil$

Proof. Notice that for every pair of consecutive vertices u, v on L_i , each of distance one to P , if u and v have descendants, then the pair u, v corresponds to some pair of descendants on L_{i+1} which contribute 1 to ex_{i+1} . Furthermore, in a right-layered tree, the vertices of L_i with distance one to P are unique as incident vertices to edges of L_i which may contribute to θ_i in repeating weights of L_{i-1} .

Notice that consecutive vertices of L_{i+1} of distance 2 away from each other, and distance 2 from P , may be incident to edges of L_i with weights that occurred on L_{i-1} . Moreover, these weights are consecutive. However, for every consecutive pair of vertices of L_{i+1} of distance more than 2 away from each other, and distance 2 from P , the weights of the corresponding edges have a difference of 2.

Also, note that the minimum weight of edges of L_{i-1} that are incident to edges of L_i cannot be repeated since such a weight is on an edge e , which may only be incident to the same vertex v as the edge f with the minimum weight of L_i , and the other vertices incident to e and f must have different labels.

Suppose next that d_i is an integer and for every vertex v of L_i of distance 1 from P with at least one neighbor not on P , $\deg(v) = d_i$. Call this the uniform case, and notice in light of the above observations, $\theta_i = \left\lceil \frac{d_i-1}{d_i} ex_{i+1} \right\rceil$. Moreover, in a right-layered tree, the degree sequence of vertices of L_i but not of P , with neighbors on L_{i+1} , are monotonically increasing so that the number of weights that are skipped on level i is the same as in the uniform case, though the skips in weights may occur between edges farther to the left. This observation completes the proof. \square

Notice that by Claim 2.21 and formula 2.6 we can write

$$\begin{aligned} \alpha(i, i+1) &\geq m(i, i+1) - \theta_{i+1} \geq m(i, i+1) - \left\lceil \frac{d_i-1}{d_i} ex_{i+1} \right\rceil \\ &\geq m(i, i+1) - \left\lceil \frac{(d_i-1)(m(i, i+1)-1)}{d_i^2} \right\rceil \end{aligned} \quad (2.7)$$

The above term is minimized when $d_i = 2$ and we obtain the bound

$$\alpha(i, i+1) \geq m(i, i+1) - \left\lceil \frac{m(i, i+1)-1}{4} \right\rceil \quad (2.8)$$

Note that $m(i-1, i)$ is odd by definition, and the above inequality can be rewritten as

$$\alpha(i, i+1) \geq \begin{cases} \frac{3}{4}m(i, i+1) + \frac{1}{4}, & \text{when } m(i, i+1) \equiv 1 \pmod{4} \\ \frac{3}{4}m(i, i+1) - \frac{1}{4}, & \text{when } m(i, i+1) \equiv 3 \pmod{4} \end{cases}$$

Furthermore, edges of distance 0 or 1 from P on levels 0, 1, and $d-1$, have weights that are never repeated.

Thus, the “worst case” for the number of distinct edge weights of any lobster T in our labeling is one where $m(i, i+1)$ is congruent to 3 (mod 4), producing the number of distinct weights of T as

$$\alpha(T) \geq 1 + \sum_{i=0}^{d-1} \alpha(i, i+1) \geq \frac{3}{4}(m-3) - \frac{1}{4}(d-3) + 3 = \frac{3m-d+6}{4}$$

Notice that this bound is an improvement on [9] for small diameter trees, in particular, when $d < \frac{m+22}{7}$. With a few modifications, we can also improve the α -size of lobsters with large diameter.

We observe that when $d_i = 2$ for all i , the number of levels with incident edges to P is at most $\frac{m-3-d}{2}$. Thus we calculate

$$\alpha(T) \geq 1 + \sum_{i=0}^{d-1} \alpha(i, i+1) \geq \frac{3}{4}(m-3) - \frac{1}{4} \left(\frac{m-3-d}{2} - 3 \right) + 3 = \frac{5m+d+3}{8}$$

This bound is an improvement on [9] when $d > \frac{5m+19}{7}$. \square

It is not difficult to find a lobster T with a perfect matching such that any BH labeling of T is not bipartite, as in the next example.

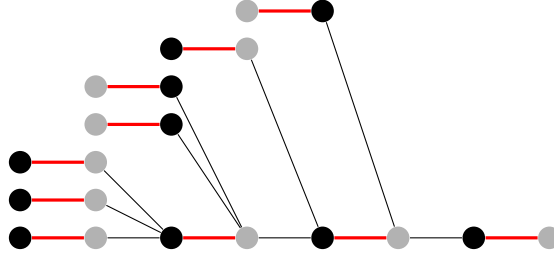


FIGURE 1. A lobster with a perfect matching but no bipartite BH labeling

Although we cannot improve bounds on $\alpha(T)$ by applying BH labelings, the gracesize is another matter.

For any lobster T of size m , let twice the size of a maximum matching on T be $\nu(T)$, or just ν if T is clear from context.

Theorem 2.22. *For any lobster T with m edges and diameter d ,*

$$gs(T) \geq \max \left\{ \frac{3}{4}m + \frac{d-\nu}{8} + \frac{3}{2}, \nu \right\}.$$

Proof. We continue from the proof of Theorem 2.18 with the same terminology and notation. Observe that T can be viewed as a lobster shell of order ν with $m - \nu$ amalgamated leaves. From this perspective, note that the number of levels of T with incident paths of length 2 which are not on P is at most $\frac{m-d-(m-\nu)}{2} = \frac{\nu-d}{2}$. Summing 2.8 over all such i , we obtain

$$gs(T) \geq \frac{3}{4}(m-3) - \frac{1}{4} \left(\frac{\nu-d}{2} - 3 \right) + 3$$

which is the first desired bound. The second bound is just Proposition 2.7. \square

Note: The bound from the above theorem implies the following improvement

Corollary 2.23.

$$\text{If } \nu \geq \frac{3}{4}m, \text{ then } gs(T) \geq \frac{3}{4}m \quad (2.9)$$

$$\text{If } \nu < \frac{3}{4}m, \text{ then } gs(T) \geq \frac{3}{4}m \text{ for } d \geq \nu - 12 \quad (2.10)$$

3. REMARKS

The improvement in the gracesize bound from Theorem 2.22 comes at the cost of the labeling not necessarily being bipartite. However, this is the first instance of the use of a non-bipartite labeling in such a result, which we view as the correct approach since the conjectured bound from the GTC could not come from a bipartite labeling. A promising direction could be to find values of d_i that produce the minimum gracesize of T simultaneously by equation 2.7 and Proposition 2.7.

Our approach shows improvements for range-relaxed graceful labeling and edge-relaxed graceful labelings of lobsters as a step towards Bermond's conjecture [1] that all lobsters are graceful. However, with more careful analysis of the excess of k -distant trees for $k > 2$, analogous statements may be possible.

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